

# ENUMERATION OF 2-POLYMATROIDS ON UP TO SEVEN ELEMENTS

THOMAS J. SAVITSKY

**ABSTRACT.** A theory of single-element extensions of integer polymatroids analogous to that of matroids is developed. We present an algorithm to generate a catalog of 2-polymatroids, up to isomorphism. When we implemented this algorithm on a computer, obtaining all 2-polymatroids on at most seven elements, we discovered the surprising fact that the number of 2-polymatroids on seven elements fails to be unimodal in rank.

## 1. INTRODUCTION

A  $k$ -polymatroid is a generalization of a matroid in which the rank of an element may be greater than 1 but cannot exceed  $k$ . Precise definitions are given in the next section. Polymatroids have applications in mathematics and computer science. For example, Chapter 11 of [6] employs 2-polymatroids in the study of matching theory. Polymatroids, and more generally, submodular functions, arise in combinatorial optimization; see Part IV of [14]. We take the perspective that  $k$ -polymatroids are worth studying in their own right.

Although much work has been done with the use of computers on the enumeration of small matroids, to our knowledge, none has been done on enumerating  $k$ -polymatroids, where  $k > 1$ . Some landmark results in matroid enumeration include the following: in 1973, Blackburn, Crapo, and Higgs [2] published a catalog of all simple matroids on at most eight elements; in 2008, Mayhew and Royle [9] produced a catalog of all matroids on up to nine elements; and in 2012, Matsumoto, Moriyama, Imai, and Bremner [7] enumerated all rank-4 matroids on ten elements.

In this paper, we describe our success in adapting the approach used by Mayhew and Royle to 2-polymatroids. Using a desktop computer, we produced a catalog of all 2-polymatroids, up to isomorphism, on at most seven elements. We were surprised to discover that the number of 2-polymatroids on seven elements is not unimodal in rank.

## 2. BACKGROUND

For an introduction to polymatroids, we recommend Chapter 12 of [13]. We begin our discussion with definitions.

**Definition 1.** Let  $S$  be a finite set. Suppose  $\rho: 2^S \rightarrow \mathbb{N}$  satisfies the following three conditions:

- (i) if  $X, Y \subseteq S$ , then  $\rho(X \cap Y) + \rho(X \cup Y) \leq \rho(X) + \rho(Y)$  (submodular),
- (ii) if  $X \subseteq Y \subseteq S$ , then  $\rho(X) \leq \rho(Y)$  (monotone), and
- (iii)  $\rho(\emptyset) = 0$  (normalized).

Then  $(\rho, S)$  is termed an integer polymatroid or simply a polymatroid with rank function  $\rho$  and ground set  $S$ .

**Definition 2.** Let  $k$  be a positive integer, and let  $(\rho, S)$  be a polymatroid. Suppose that  $\rho(x) \leq k$  for every  $x \in S$ . Then  $(\rho, S)$  is a  $k$ -polymatroid. A matroid may be defined as a 1-polymatroid.

Let  $(\rho, S)$  and  $(\tau, T)$  be polymatroids. A function  $\sigma: S \rightarrow T$  is an *isomorphism* of polymatroids if  $\sigma$  is a bijection and if  $\rho(X) = \tau(\sigma(X))$  for every  $X \subseteq S$ . The closure operator of a polymatroid may be defined exactly as that of a matroid.

**Definition 3.** The closure operator  $\text{cl}: 2^S \rightarrow 2^S$  of a polymatroid  $(\rho, S)$  is given by  $\text{cl}_\rho(X) = \{x : \rho(X \cup x) = \rho(X)\}$  for  $X \subseteq S$ . The set  $\text{cl}_\rho(X)$  is called the closure of  $X$  with respect to  $\rho$ . The subscript is omitted when  $\rho$  is clear from context.

One can show that  $\rho(X) = \rho(\text{cl}(X))$  by induction on  $|\text{cl}(X) - X|$ . We will freely make use of this as well as the following properties of closure operators. They are stated here without proof.

**Proposition 4.** *The closure operator of a polymatroid  $(\rho, S)$  satisfies the following three properties:*

- (i)  $X \subseteq \text{cl}(X)$  for all  $X \subseteq S$  (increasing),
- (ii) if  $X \subseteq Y \subseteq S$ , then  $\text{cl}(X) \subseteq \text{cl}(Y)$  (monotone), and
- (iii)  $\text{cl}(X) = \text{cl}(\text{cl}(X))$  for all  $X \subseteq S$  (idempotent).

A subset of the ground set that is maximal with respect to rank is called a *flat*. Here is the definition in terms of the closure operator.

**Definition 5.** *Let  $(\rho, S)$  be a polymatroid. A set  $X \subseteq S$  is called a flat of  $\rho$  if  $\text{cl}(X) = X$ . The collection of flats of  $(\rho, S)$  is symbolized by  $\mathcal{F}(\rho, S)$ .*

Intersections of flats of matroids are themselves flats, and the same is true for polymatroids.

**Proposition 6.** *If  $F$  and  $G$  are flats of polymatroid  $(\rho, S)$ , then  $F \cap G$  is also a flat.*

*Proof.* Let  $x \in S - (F \cap G)$ . Either  $x \in S - F$  or  $x \in S - G$ . By relabeling  $F$  and  $G$  if necessary, we may assume  $x \in S - F$ . By submodularity,

$$\rho(F) + \rho((F \cap G) \cup x) \geq \rho(F \cup x) + \rho(F \cap G).$$

This implies  $\rho((F \cap G) \cup x) - \rho(F \cap G) \geq \rho(F \cup x) - \rho(F)$ . By assumption,  $\rho(F \cup x) - \rho(F) > 0$ , and hence, as needed,  $\rho((F \cap G) \cup x) - \rho(F \cap G) > 0$ .  $\square$

Since the entire ground set of a polymatroid is a flat, we see that the collection of flats of a polymatroid forms a lattice under set-inclusion.

The theory of single-element extensions of matroids was developed by Crapo in [3]. We extend this theory to polymatroids in the next section, but first the matroid case is briefly reviewed here. See Section 7.2 of [13] for a detailed exposition. We begin with a couple of definitions that apply to polymatroids as well.

**Definition 7.** *Let  $(\rho, S)$  be a polymatroid, and let  $e$  be an element not in  $S$ . If  $(\bar{\rho}, S \cup e)$  is a polymatroid with  $\bar{\rho}(X) = \rho(X)$  for all  $X \subseteq S$ , then  $\bar{\rho}$  is a single-element extension of  $\rho$ .*

**Definition 8.** *A modular cut of a polymatroid  $(\rho, S)$  is a subset  $\mathcal{M}$  of  $\mathcal{F}(\rho, S)$  for which*

- (i) if  $F \in \mathcal{M}$ ,  $G \in \mathcal{F}(\rho, S)$ , and  $F \subseteq G$ , then  $G \in \mathcal{M}$ , and
- (ii) if  $F, G \in \mathcal{M}$  and  $\rho(F \cap G) + \rho(F \cup G) = \rho(F) + \rho(G)$ , then  $F \cap G \in \mathcal{M}$ .

The next two results show that single-element extensions of a matroid can be placed in one-to-one correspondence with its modular cuts. This correspondence underlies the enumeration efforts in [2] and [9].

**Theorem 9.** *Suppose  $(r, S)$  is a matroid with single-element extension  $(\bar{r}, S \cup e)$ . Define  $\mathcal{M} = \{F \in \mathcal{F}(r, S) : r(F) = \bar{r}(F \cup e)\}$ . Then  $\mathcal{M}$  is a modular cut.*

**Theorem 10.** *Suppose  $(r, S)$  is a matroid,  $e$  is an element not in  $S$ , and  $\mathcal{M} \subseteq \mathcal{F}(r, S)$  is a modular cut. Define  $\bar{r} : 2^{S \cup e} \rightarrow \mathbb{N}$  as follows: for  $X \subseteq S$ , set  $\bar{r}(X) = r(X)$  and*

$$\bar{r}(X \cup e) = \begin{cases} r(X) & \text{if } \text{cl}(X) \in \mathcal{M}, \\ r(X) + 1 & \text{otherwise.} \end{cases}$$

*Then  $(\bar{r}, S \cup e)$  is a matroid and a single-element extension of  $(r, S)$ .*

Our final definition in this section will be used when we describe the flats of single-element extensions.

**Definition 11.** *Let  $F$  and  $G$  be flats of a polymatroid  $(\rho, S)$ . Suppose that  $F \subsetneq G$  and that for any flat  $H$  with  $F \subseteq H \subseteq G$ , either  $H = F$  or  $H = G$ . Then we say that  $G$  covers  $F$ .*

## 3. SINGLE-ELEMENT EXTENSIONS OF POLYMATROIDS

Given a polymatroid, our aim is to describe all of its single-element extensions. As in the matroid case we may restrict our attention to flats of the original polymatroid. Suppose  $(\bar{\rho}, S \cup e)$  is a single-element extension of  $(\rho, S)$ . The following proposition shows that if the value of  $\bar{\rho}(F \cup e)$  is known for every flat  $F$  of  $(\rho, S)$ , then  $\bar{\rho}$  is completely determined.

**Proposition 12.** *Suppose  $(\bar{\rho}, S \cup e)$  is a single-element extension of  $(\rho, S)$ . Let  $X \subseteq S$ , and let  $\text{cl}(X)$  be the closure of  $X$  with respect to  $\rho$  (not  $\bar{\rho}$ ). Then  $\bar{\rho}(X \cup e) = \bar{\rho}(\text{cl}(X) \cup e)$ .*

*Proof.* Since  $X \cup e \subseteq \text{cl}(X) \cup e = \text{cl}_{\bar{\rho}}(X) \cup e \subseteq \text{cl}_{\bar{\rho}}(X \cup e)$  and  $\bar{\rho}$  has the same value on the first and last of these sets, the result follows.  $\square$

For a single-element extension  $(\bar{\rho}, S \cup e)$  of  $(\rho, S)$ , let  $c$  be  $\bar{\rho}(e)$  and let  $X \subseteq S$ . It follows that  $\bar{\rho}(X \cup e) \leq \rho(X) + c$  by the submodularity and normalization of  $\bar{\rho}$ . Therefore, we may partition the flats of  $(\rho, S)$  into classes  $\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_c$  by the rule  $F \in \mathcal{M}_i$  if and only if  $\bar{\rho}(F \cup e) = \rho(F) + i$ . (Note that some  $\mathcal{M}_i$  may be empty.) By Proposition 12, knowledge of  $(\rho, S)$  and the partition  $(\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_c)$  completely determines  $(\bar{\rho}, S \cup e)$ . Our goal is to develop properties that characterize such partitions. The following definition will be useful.

**Definition 13.** *Let  $(\rho, S)$  be a polymatroid, and let  $X, Y \subseteq S$ . Define the modular defect of  $X$  and  $Y$ , denoted  $\delta(X, Y)$ , to be  $\rho(X) + \rho(Y) - \rho(X \cup Y) - \rho(X \cap Y)$ . If  $\delta(X, Y) = 0$ , then  $X$  and  $Y$  are a modular pair of sets.*

Now suppose  $(\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_c)$  is a partition of  $\mathcal{F}(\rho, S)$ . Let  $e$  be an element not in  $S$  and define  $\bar{\rho}: 2^{S \cup e} \rightarrow \mathbb{N}$  as follows: for  $X \subseteq S$ , set  $\bar{\rho}(X) = \rho(X)$  and, if  $\text{cl}(X) \in \mathcal{M}_i$ , then set  $\bar{\rho}(X \cup e) = \rho(X) + i$ . Furthermore, define a function  $\mu: 2^S \rightarrow \mathbb{N}$  by  $\mu(X) = i$  if  $\text{cl}(X) \in \mathcal{M}_i$ .

**Theorem 14.** *As defined above,  $(\bar{\rho}, S \cup e)$  is a polymatroid, and hence a single-element extension of  $(\rho, S)$ , if and only if the following three conditions hold for all flats  $F, G$  of  $(\rho, S)$ :*

- (I)  $\mu(F \cap G) + \mu(F \cup G) - \delta(F, G) \leq \mu(F) + \mu(G)$ ,
- (II) if  $F \subseteq G$ , then  $\rho(F) + \mu(F) \leq \rho(G) + \mu(G)$ , and
- (III) if  $F \subseteq G$ , then  $\mu(G) \leq \mu(F)$ .

*Proof.* Assume  $(\bar{\rho}, S \cup e)$  is a polymatroid, and let  $F, G$  be flats of  $(\rho, S)$ . Applying the submodularity of  $\bar{\rho}$  to the pair of sets  $F \cup e$  and  $G \cup e$  gives

$$\bar{\rho}((F \cup e) \cap (G \cup e)) + \bar{\rho}((F \cup e) \cup (G \cup e)) \leq \bar{\rho}(F \cup e) + \bar{\rho}(G \cup e).$$

By our definition of  $\bar{\rho}$ , the right side of the above inequality equals  $\rho(F) + \mu(F) + \rho(G) + \mu(G)$ . The left side equals

$$\begin{aligned} \bar{\rho}((F \cap G) \cup e) + \bar{\rho}((F \cup G) \cup e) &= \rho(F \cap G) + \mu(F \cap G) + \rho(F \cup G) + \mu(F \cup G) \\ &= \mu(F \cap G) + \mu(F \cup G) + \rho(F) + \rho(G) - \delta(F, G). \end{aligned}$$

We conclude that  $\mu(F \cap G) + \mu(F \cup G) - \delta(F, G) \leq \mu(F) + \mu(G)$  and see that condition (I) is satisfied.

Statement (II) is the monotone property of  $\bar{\rho}$ .

Finally, to show condition (III), apply the submodularity of  $\bar{\rho}$  to the pair of sets  $F \cup e$  and  $G$ . This gives the first of the following equivalent inequalities:

- (1)  $\bar{\rho}((F \cup e) \cup G) + \bar{\rho}((F \cup e) \cap G) \leq \bar{\rho}(F \cup e) + \bar{\rho}(G)$
- (2)  $\bar{\rho}(G \cup e) + \bar{\rho}(F) \leq \bar{\rho}(F \cup e) + \bar{\rho}(G)$
- (3)  $\bar{\rho}(G \cup e) - \bar{\rho}(G) \leq \bar{\rho}(F \cup e) - \bar{\rho}(F)$
- (4)  $\mu(G) \leq \mu(F)$ .

Now assume that conditions (I), (II), and (III) are satisfied. We must verify that  $\bar{\rho}$  satisfies the three axioms for a polymatroid. It follows immediately from our definition that  $\bar{\rho}(\emptyset) = 0$ .

Next, we check monotonicity. Assume that  $X \subseteq Y \subseteq S$ . The definition of  $\bar{\rho}$  and the monotonicity of  $\rho$  imply that  $\bar{\rho}(X) = \rho(X) \leq \rho(Y) = \bar{\rho}(Y)$ . Thus we also get  $\bar{\rho}(X) \leq \bar{\rho}(Y) \leq \bar{\rho}(Y \cup e)$ . It remains to check

that  $\bar{\rho}(X \cup e) \leq \bar{\rho}(Y \cup e)$ . Observe

$$\begin{aligned}
 \bar{\rho}(X \cup e) &= \rho(X) + \mu(X) \\
 &= \rho(\text{cl}(X)) + \mu(\text{cl}(X)) \\
 &\leq \rho(\text{cl}(Y)) + \mu(\text{cl}(Y)) \quad (\text{by condition (II)}) \\
 &= \rho(Y) + \mu(Y) \\
 &= \bar{\rho}(Y \cup e).
 \end{aligned}$$

Therefore,  $\bar{\rho}$  is monotone on all subsets of  $S \cup e$ .

Since  $\bar{\rho}(X) = \rho(X)$  for  $X \subseteq S$ , to check submodularity it suffices to verify it for the pairs (a)  $X \cup e$  and  $Y$ , and (b)  $X \cup e$  and  $Y \cup e$ , with  $X, Y \subseteq S$ . For case (a), we have

$$\begin{aligned}
 \bar{\rho}((X \cup e) \cap Y) + \bar{\rho}((X \cup e) \cup Y) &= \bar{\rho}(X \cap Y) + \bar{\rho}((X \cup Y) \cup e) \\
 &= \rho(X \cap Y) + \rho(X \cup Y) + \mu(\text{cl}(X \cup Y)) \\
 &\leq \rho(X) + \rho(Y) + \mu(\text{cl}(X \cup Y)) \quad (\text{by the submodularity of } \rho) \\
 &\leq \rho(X) + \rho(Y) + \mu(\text{cl}(X)) \quad (\text{by condition (III)}) \\
 &= \bar{\rho}(X \cup e) + \bar{\rho}(Y).
 \end{aligned}$$

For case (b), we have

$$\begin{aligned}
 \bar{\rho}(X \cup e) + \bar{\rho}(Y \cup e) &= \rho(\text{cl}(X)) + \mu(\text{cl}(X)) + \rho(\text{cl}(Y)) + \mu(\text{cl}(Y)) \\
 &\geq \mu(\text{cl}(X) \cap \text{cl}(Y)) + \mu(\text{cl}(X) \cup \text{cl}(Y)) - \delta(\text{cl}(X), \text{cl}(Y)) + \rho(\text{cl}(X)) + \rho(\text{cl}(Y)) \\
 &= \mu(\text{cl}(X) \cap \text{cl}(Y)) + \mu(\text{cl}(X) \cup \text{cl}(Y)) + \rho(\text{cl}(X) \cup \text{cl}(Y)) + \rho(\text{cl}(X) \cap \text{cl}(Y)) \\
 &= \bar{\rho}((\text{cl}(X) \cup \text{cl}(Y)) \cup e) + \bar{\rho}((\text{cl}(X) \cap \text{cl}(Y)) \cup e) \\
 &\geq \bar{\rho}(X \cup Y \cup e) + \bar{\rho}((X \cap Y) \cup e).
 \end{aligned}$$

The first inequality follows by condition (I), and the last inequality holds because the monotonicity of  $\bar{\rho}$  has already been established.  $\square$

Note that Theorem 14 generalizes Theorems 9 and 10 for single-element extensions of matroids. Also note that if the conditions of the theorem are satisfied, then  $\mathcal{M}_0$  is a modular cut. Lastly, we point out that the theorem remains true if the word “flats” is replaced by “sets” in its statement.

**Definition 15.** A partition  $(\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_c)$  of flats of a polymatroid  $(\rho, S)$  that satisfies the conditions in Theorem 14 is called an extensible partition.

For the remainder of this section, assume that  $(\rho, S)$  is a polymatroid,  $(\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_c)$  is an extensible partition, and  $(\bar{\rho}, S \cup e)$  is the single-element extension defined right before Theorem 14. Our next goal is to describe the flats of  $(\bar{\rho}, S \cup e)$ .

Clearly if  $F \subseteq S$  is a flat of  $(\bar{\rho}, S \cup e)$ , then  $F$  is also a flat of  $(\rho, S)$ . We also have the following helpful fact.

**Proposition 16.** For  $F \subseteq S$ , if  $F \cup e$  is a flat of  $(\bar{\rho}, S \cup e)$ , then  $F$  is a flat of  $(\rho, S)$ .

*Proof.* Observe that  $\text{cl}_\rho(F) \subseteq \text{cl}_{\bar{\rho}}(F) \subseteq \text{cl}_{\bar{\rho}}(F \cup e) = F \cup e$ .  $\square$

Therefore, to find the flats of  $(\bar{\rho}, S \cup e)$  we need only consider sets of the form  $F$  and  $F \cup e$ , where  $F$  is a flat of  $(\rho, S)$ . The next proposition explicitly describes the flats of  $(\bar{\rho}, S \cup e)$ .

**Proposition 17.** Let  $(\bar{\rho}, S \cup e)$  be the single-element extension of  $(\rho, S)$  corresponding to the extensible partition  $(\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_c)$ . The flats of  $(\bar{\rho}, S \cup e)$  are the sets

- (1)  $F$  in  $\mathcal{M}_i$ , for  $i > 0$ ,
- (2)  $F \cup e$ , for  $F \in \mathcal{M}_0$ ,
- (3)  $F \cup e$ , for  $F \in \mathcal{M}_i$  with  $i > 0$ , where  $F$  has no cover  $G$  with  $\rho(F) + \mu(F) = \rho(G) + \mu(G)$ .

*Proof.* To reiterate, we need only look at sets of the form  $F$  and  $F \cup e$ , where  $F$  is a flat of  $(\rho, S)$ .

It follows from the definition of  $\bar{\rho}$  that a flat  $F$  of  $(\rho, S)$  is a flat of  $(\bar{\rho}, S \cup e)$  if and only if  $F \notin \mathcal{M}_0$ .

If  $F \in \mathcal{M}_0$ , then  $F \cup e$  is a flat of  $(\bar{\rho}, S \cup e)$  since, for  $y \in S - F$ , we have

$$\bar{\rho}(F \cup \{e, y\}) \geq \rho(F \cup y) > \rho(F) = \bar{\rho}(F \cup e).$$

We claim that for  $F \in \mathcal{M}_i$  with  $i > 0$ , the set  $F \cup e$  is a flat of  $(\bar{\rho}, S \cup e)$  if and only if the inequality in property (II) of Theorem 14 is strict for all covers  $G$  of  $F$ . Indeed, if  $G$  covers  $F$  and  $\rho(F) + \mu(F) = \rho(G) + \mu(G)$ , then  $F \cup e$  is not a flat since

$$\bar{\rho}(F \cup e) = \rho(F) + \mu(F) = \rho(G) + \mu(G) = \bar{\rho}(G \cup e).$$

Now assume strict inequality holds in property (II) for all covers of  $F$ . If  $x \in S - F$ , then there is a cover  $G$  of  $F$  with  $F \subsetneq G \subseteq \text{cl}_\rho(F \cup x)$ , so

$$\bar{\rho}(F \cup e) = \rho(F) + \mu(F) < \rho(G) + \mu(G) = \bar{\rho}(G \cup e) \leq \bar{\rho}(F \cup \{e, x\}). \quad \square$$

Note that if  $\mu(G) = \mu(F)$ , then equality cannot hold in property (II) of Theorem 14, since  $\rho(G) > \rho(F)$ .

These results generalize those for matroid extension. We define the *collar* of  $\mathcal{M}_i$  to consist of every  $F \in \mathcal{M}_i$  that is covered by some  $G \in \mathcal{M}_j$  with  $j < i$ . In a matroid  $(r, S)$ , if a flat  $G$  covers a flat  $F$ , then  $r(G) - r(F) = 1$ . If  $(\bar{r}, S \cup e)$  is a single-element extension and  $F \in \mathcal{M}_1$ , then  $F \cup e$  is a flat of  $\bar{r}$  if and only if  $F$  is not in the collar of  $\mathcal{M}_1$ .

#### 4. GENERATING A CATALOG OF SMALL 2-POLYMATROIDS

Now we will specialize the results of the previous section to 2-polymatroids.

Suppose  $(\rho, S)$  is a 2-polymatroid with collection of flats  $\mathcal{F}(S)$ . Suppose that  $\mathcal{F}(S)$  is the union of three disjoint sets,  $\mathcal{M}_0$ ,  $\mathcal{M}_1$ , and  $\mathcal{M}_2$ , some of which may be empty. Let  $e$  be an element not in  $S$ . We define a function  $\bar{\rho}: 2^{S \cup e} \rightarrow \mathbb{N}$  as follows. For  $X \subseteq S$ , define  $\bar{\rho}(X) = \rho(X)$  and

$$\bar{\rho}(X \cup e) = \rho(X) + i \text{ where } \text{cl}(X) \in \mathcal{M}_i.$$

When computing the extensible partitions of a 2-polymatroid, we found it convenient to work with the following verbose specialization of Theorem 14.

**Theorem 18.** *As defined,  $(\bar{\rho}, S \cup e)$  is a 2-polymatroid extension of  $(\rho, S)$  if and only if the following seven conditions are met.*

- (1) *If  $F \in \mathcal{M}_2$ ,  $G \in \mathcal{F}(S)$ ,  $F \subseteq G$ , and  $\rho(G) - \rho(F) = 1$ , then  $G \in \mathcal{M}_1 \cup \mathcal{M}_2$ . In other words, if  $F \in \mathcal{M}_2$  is covered by a flat  $G$  of  $(\rho, S)$  of one rank higher, then  $G$  cannot be in  $\mathcal{M}_0$ .*
- (2) *If  $F, G \in \mathcal{M}_0$  and  $(F, G)$  is a modular pair, then  $F \cap G \in \mathcal{M}_0$  as well.*
- (3) *If  $F, G \in \mathcal{M}_0$  and  $\rho(F) + \rho(G) = \rho(F \cup G) + \rho(F \cap G) + 1$ , then  $F \cap G \in \mathcal{M}_0 \cup \mathcal{M}_1$ .*
- (4) *If  $F, G \in \mathcal{M}_1$  and  $(F, G)$  is a modular pair, then either  $F \cap G \in \mathcal{M}_1$  as well, or  $F \cap G \in \mathcal{M}_2$  and  $\text{cl}(F \cup G) \in \mathcal{M}_0$ .*
- (5) *If  $F \in \mathcal{M}_0$ ,  $G \in \mathcal{M}_1$ , and  $(F, G)$  is a modular pair, then  $F \cap G$  cannot be in  $\mathcal{M}_2$ .*
- (6) *The set  $\mathcal{M}_2$  is down-closed in the lattice  $\mathcal{F}(\rho, S)$ .*
- (7) *The set  $\mathcal{M}_0$  is up-closed in the lattice  $\mathcal{F}(\rho, S)$ .*

*Sketch of Proof.* Condition (II) of Theorem 14 specializes to condition (1) here, condition (I) to conditions (2) through (5), and condition (III) to conditions (6) and (7).  $\square$

The flats of  $(\bar{\rho}, S \cup e)$  are the sets

- (1)  $F$  in  $\mathcal{M}_1 \cup \mathcal{M}_2$ ,
- (2)  $F \cup e$ , for  $F \in \mathcal{M}_0$ ,
- (3)  $F \cup e$ , for  $F \in \mathcal{M}_i$ , with  $i > 0$ , where,
  - (a)  $F$  has no cover  $G$  in  $\mathcal{M}_{i-1}$  with  $\rho(G) = \rho(F) + 1$ , and
  - (b) if  $i = 2$ ,  $F$  has no cover  $G$  in  $\mathcal{M}_0$  with  $\rho(G) = \rho(F) + 2$ .

For example, let  $(\rho, \{a, b\})$  be the 2-polymatroid consisting of two lines placed freely in a plane. To be specific, define  $\rho(\emptyset) = 0$ ,  $\rho(\{a\}) = \rho(\{b\}) = 2$ , and  $\rho(\{a, b\}) = 3$ . The single-element extension corresponding to the extensible partition

$$(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2) = (\{\{a, b\}\}, \{\{a\}, \{b\}\}, \{\emptyset\})$$

is the 2-polymatroid consisting of three lines placed freely in a plane.

Using the results of this section, we endeavored to catalog all small 2-polymatroids on a computer by means of a canonical deletion algorithm.

**Definition 19.** Suppose  $\mathcal{X}$  is a collection of combinatorial objects with ground set  $\{1, \dots, n\}$  and a notion of isomorphism. A function  $C: \mathcal{X} \rightarrow \mathcal{X}$  is a canonical labeling function if the following hold for all  $X, Y \in \mathcal{X}$ :

- (i)  $X$  is isomorphic to  $C(X)$ , and
- (ii)  $C(X) = C(Y)$  if and only if  $X$  is isomorphic to  $Y$ .

In this case,  $C(X)$  is called the canonical representative of  $X$ .

Brendan McKay's `nauty` program efficiently computes canonically labelings of colored graphs. In order to make use of it, we convert polymatroids into graphs using the following construction.

**Definition 20.** Given an integer polymatroid  $(\rho, S)$ , define a colored, bipartite graph with bipartition  $S$  and  $\mathcal{F}(\rho, S)$ . An edge between  $e \in S$  and  $F \in \mathcal{F}(\rho, S)$  exists if and only if  $e \in F$ . Color  $F \in \mathcal{F}(\rho, S)$  with its rank,  $\rho(F)$ . Color each  $e \in S$  with  $-1$ . Call the resulting graph the flat graph<sup>1</sup> of the integer polymatroid.

Note that if  $X \subseteq S$  and if  $F$  is the smallest flat containing  $X$ , then  $\rho(X) = \rho(F)$ . In terms of the flat graph, the rank of a set  $X \subseteq S$  equals the least color amongst those vertices adjacent to every element of  $X$ . Using this observation, it is easy to prove the next proposition.

**Proposition 21.** Two integer polymatroids are isomorphic if and only if their flat graphs are isomorphic as colored graphs. (By an isomorphism of a colored graph, we mean a graph isomorphism that maps each vertex to another of the same color.)

Therefore, in order to canonically label a 2-polymatroid, it suffices to consider its flat graph. Then `nauty` is used to compute a canonical labeling of the flat graph. When restricted to the ground set of the polymatroid, this gives a canonical labeling of the polymatroid. For a description of the algorithms used by `nauty` see [10] and [11]. One may also find the exposition in [5] helpful.

Now we have all the tools needed to adapt Algorithm 1 of [9] to 2-polymatroids. Suppose we are given a set  $X_n$  that consists of precisely one representative of each isomorphism class of 2-polymatroids on the ground set  $\{1, \dots, n\}$ . The following algorithm produces its counterpart,  $X_{n+1}$ , for the set  $\{1, \dots, n+1\}$ .

---

**Algorithm 1** Isomorph-free generation of 2-polymatroids

---

```

for each  $\rho \in X_n$  do
  Set  $Y_\rho \leftarrow \emptyset$ , the collection of extensions of  $\rho$  that should appear in  $X_{n+1}$ .
  for each extensible partition  $(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)$  of  $\rho$  do
    Let  $\bar{\rho}$  be the extension of  $\rho$  associated with this partition.
    Canonically label  $\bar{\rho}$ .
    Set  $\rho' \leftarrow \bar{\rho} \setminus (n+1)$ , the canonical deletion.
    Canonically label  $\rho'$ .
    if  $\rho = \rho'$  and  $\bar{\rho} \notin Y_\rho$  then
      Set  $Y_\rho \leftarrow Y_\rho \cup \bar{\rho}$ .
    end if
  end for
  Set  $X_{n+1} \leftarrow X_{n+1} \cup Y_\rho$ .
end for
return  $X_{n+1}$ 

```

---

<sup>1</sup>In our implementation, we found it prudent to insert an isolated vertex of rank  $r$  if no flats of rank  $r$  existed, for  $r < \rho(S)$ . This made it easier to work with the labelings used by `nauty`.



A few comments are in order. Note that the test  $\rho = \rho'$  is for equality, not isomorphism. In our implementation, the collections  $Y_\rho$  are binary trees, rather than merely sets, in order to speed up the search  $\rho \in Y_\rho$ .

The task of finding all extensible partitions for a polymatroid  $\rho$  is relatively straightforward, but tedious. First, a candidate for a modular cut  $\mathcal{M}_0$  is found. Since  $\mathcal{M}_0$  is an up-closed set, it suffices to keep track of the minimal flats in  $\mathcal{M}_0$ . These are found as independent sets of a graph with vertex set equal to the flats of  $\rho$ . If one flat is contained in another, an edge is placed between the two. Condition (2) of Theorem 18 is then used to narrow the search. An edge is also placed between any two flats that form a modular pair. The independent sets in this graph are the minimal members of our candidates for  $\mathcal{M}_0$ . Given an acceptable candidate for  $\mathcal{M}_0$ , a more complicated procedure is used to find all possible candidates for  $\mathcal{M}_1$ . The remaining flats are obviously assigned to  $\mathcal{M}_2$ . Unfortunately, the resulting partition must be checked to see if it satisfies conditions (1) through (5), since some of these may fail for non-minimal members of  $\mathcal{M}_0$  or  $\mathcal{M}_1$ .

Finally, note that each iteration of the outermost for loop may be run in parallel since extensions of two different members of  $X_n$  are never directly compared to each other.

## 5. IMPLEMENTATION AND RESULTS

We implemented this algorithm in the C programming language. In order to determine the cover relations for flats, we employed the ATLAS library [16] to multiply the adjacency matrices of graphs. We used the `igraph` library [4] to find independent sets in graphs. A computer with a single 6-core Intel i7-3930K processor clocked at 3.20GHz running 64-bit Ubuntu Linux executed the resulting program. After approximately four days, a catalog of all 2-polymatroids on seven or fewer elements was generated.

The following table lists the number of 2-polymatroids, up to isomorphism, on the ground set  $\{1, \dots, n\}$ , by rank.

		The number of unlabeled 2-polymatroids							
rank \ $n$		0	1	2	3	4	5	6	7
0		1	1	1	1	1	1	1	1
1			1	2	3	4	5	6	7
2				1	4	10	21	39	68
3					2	12	49	172	573
4						1	10	78	5236
5							3	49	778
6								1	21
7									4
8									
9									
10									
11									
12									
13									
14									
total		1	3	10	40	228	2380	94495	320863387

The following proposition is the key to producing the analogous table for labeled 2-polymatroids.

**Proposition 22.** *The automorphisms of an integer polymatroid  $(\rho, S)$  are in one-to-one correspondence with the automorphisms of its flat graph.*

*Sketch of Proof.* This is not hard to show. It follows, for example, from the remarks in Section 1 of [12], which employs the language of hypergraphs.  $\square$

Since `nauty` can easily compute the automorphism groups of the flat graphs of these polymatroids, applying the Orbit-Stabilizer Theorem gives a count of the number of labeled 2-polymatroids on 7 elements. The following table lists the number of labeled 2-polymatroids, on the ground set  $\{1, \dots, n\}$ , by rank.

		The number of labeled 2-polymatroids							
rank \ $n$		0	1	2	3	4	5	6	7
0		1	1	1	1	1	1	1	1
1			1	3	7	15	31	63	127
2				1	6	29	135	642	3199
3					3	41	477	5957	87477
4						1	29	784	27375
5							7	477	41695
6								1	135
7									15
8									
9									
10									
11									
12									
13									
14									
total		1	3	14	115	2040	109707	39445994	1560089623047

The symmetry of the columns in the above tables is explained by the following notion of duality for  $k$ -polymatroids.

**Definition 23.** Given a polymatroid  $(\rho, S)$ , define the  $k$ -dual  $\rho^*: 2^S \rightarrow \mathbb{N}$  by

$$\rho^*(X) = k|X| + \rho(S - X) - \rho(S).$$

It is easily seen that  $\rho^*$  is itself a  $k$ -polymatroid and that the operation of  $k$ -duality is an involution on the set of  $k$ -polymatroids on a fixed ground set which respects isomorphism. (In fact, it is shown in [17] to be the unique such involution that interchanges deletion and contraction.)

Welsh conjectured that the number of matroids on a fixed set is unimodal in rank in [15]. The counterpart of this conjecture for  $k$ -polymatroids is false. The table above shows that it fails for 2-polymatroids on 7 elements.

Since the number of labeled 2-polymatroids on seven elements is nearly a factor of  $7!$  more than the number of unlabeled ones, it seems reasonable to conjecture that, asymptotically, almost all 2-polymatroids are asymmetric.

The proof in [8] that almost all matroids are loopless carries over without change to 2-polymatroids. Our catalog suggests that a stronger property holds for 2-polymatroids. We conjecture that, asymptotically, almost all 2-polymatroids contain no elements of rank less than 2. Here is the evidence from our catalog: the number of *unlabeled* 2-polymatroids on  $\{1, \dots, n\}$  with no elements of rank less than 2.

$n$	1	2	3	4	5	6	7
count	1	2	8	51	696	49121	304541846

This table should be compared to the first table in this section.

## 6. A CONFIRMATION

Consider that the *labeled* single-element extensions of a  $k$ -polymatroid are in fact solutions to a certain integer programming problem. When all subsets of the ground set are taken as variables, inequalities guaranteeing the axioms of a  $k$ -polymatroid are easily written. To be concrete, let  $\rho: S \rightarrow \mathbb{N}$  be a  $k$ -polymatroid and let  $e$  be an element not in  $S$ . Regard  $\bar{\rho}(X)$  as a variable for each  $X \subseteq S \cup e$ . Fix  $\bar{\rho}(A) = \rho(A)$  for  $A \subseteq S$ . Also fix  $\bar{\rho}(S \cup e) = \rho(S) + c$ , where  $c$  is a natural number no greater than  $k$ . Now nonnegative integer solutions to the system of inequalities below are in one-to-one correspondence with labeled single-element extensions of  $\rho$  which increase the rank of  $\rho$  by  $c$ .



$$\begin{aligned}\bar{\rho}(A) + \bar{\rho}(A \cup f \cup g) &\leq \bar{\rho}(A \cup f) + \bar{\rho}(A \cup g) \quad \text{for } A \subseteq S \cup e \text{ and } f, g \in (S \cup e) - A; \\ 0 \leq \bar{\rho}(A \cup f) - \bar{\rho}(A) &\leq k \quad \text{for } A \subseteq S \cup e \text{ and } f \in (S \cup e) - A; \text{ and} \\ \bar{\rho}(A) &\leq k|A| \quad \text{for } A \subseteq S \cup e.\end{aligned}$$

Here, we are using a condition equivalent to submodularity; see Theorem 44.2 of [14] for a proof of equivalence. The open-source optimization software SCIP [1] is able to count the number of integer solutions to such inequalities. Using SCIP we verified the numbers of labeled 2-polymatroids given earlier. Note that, in the version of SCIP we used in April 2013, it was necessary to turn off all pre-solving options in order to obtain accurate results. This process took approximately 13 weeks using the computer described in the previous section.

## 7. ACKNOWLEDGEMENTS

The author would like to thank the anonymous referee for carefully reading this paper and providing suggestions which improved its exposition.

## REFERENCES

- [1] Tobias Achterberg. Scip: Solving constraint integer programs. *Mathematical Programming Computation*, 1(1):1–41, July 2009.
- [2] John E. Blackburn, Henry H. Crapo, and Denis A. Higgs. A catalogue of combinatorial geometries. *Math. Comp.*, 27:155–166; addendum, *ibid.* 27 (1973), no. 121, loose microfiche suppl. A12–G12, 1973.
- [3] Henry H. Crapo. Single-element extensions of matroids. *J. Res. Nat. Bur. Standards Sect. B*, 69B:55–65, 1965.
- [4] Gábor Csárdi and Tamás Nepusz. The igraph software package for complex network research. *InterJournal Complex Systems*, page 1695, 2006.
- [5] Stephen G. Hartke and A. J. Radcliffe. McKay’s canonical graph labeling algorithm. In *Communicating Mathematics*, volume 479 of *Contemp. Math.*, pages 99–111. Amer. Math. Soc., Providence, RI, 2009.
- [6] László Lovász and Michael D. Plummer. *Matching Theory*. AMS Chelsea Publishing, Providence, RI, 2009. Corrected reprint of the 1986 original.
- [7] Yoshitake Matsumoto, Sonoko Moriyama, Hiroshi Imai, and David Bremner. Matroid enumeration for incidence geometry. *Discrete Comput. Geom.*, 47(1):17–43, 2012.
- [8] Dillon Mayhew, Mike Newman, Dominic Welsh, and Geoff Whittle. On the asymptotic proportion of connected matroids. *European J. Combin.*, 32(6):882–890, 2011.
- [9] Dillon Mayhew and Gordon F. Royle. Matroids with nine elements. *J. Combin. Theory Ser. B*, 98(2):415–431, 2008.
- [10] Brendan D. McKay. Practical graph isomorphism. In *Proceedings of the Tenth Manitoba Conference on Numerical Mathematics and Computing, Vol. I (Winnipeg, Man., 1980)*, volume 30, pages 45–87, 1981.
- [11] Brendan D. McKay and Adolfo Piperno. Practical graph isomorphism, II. *J. Symbolic Comput.*, 60:94–112, 2014.
- [12] Gary L. Miller. Isomorphism of graphs which are pairwise  $k$ -separable. *Inform. and Control*, 56(1-2):21–33, 1983.
- [13] James Oxley. *Matroid Theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.
- [14] Alexander Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. B*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Matroids, trees, stable sets, Chapters 39–69.
- [15] D. J. A. Welsh. Combinatorial problems in matroid theory. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, pages 291–306. Academic Press, London, 1971.
- [16] R. Clint Whaley and Antoine Petit. Minimizing development and maintenance costs in supporting persistently optimized BLAS. *Software: Practice and Experience*, 35(2):101–121, February 2005.
- [17] Geoff Whittle. Duality in polymatroids and set functions. *Combin. Probab. Comput.*, 1(3):275–280, 1992.